

COHOMOLOGY OBSTRUCTIONS TO CERTAIN CIRCLE ACTIONS

PING LI

ABSTRACT. Given a closed oriented manifold M^{2n} and $c \in H^2(M)$, we shall investigate the relations between the non-vanishing of $(c^n) \cdot [M]$ and lower bounds of the fixed points of circle actions on M . We first review some known results, due to Hattori, Fang-Rong and Pelayo-Tolman respectively, then unify and generalize these results into a new theorem. The main ingredients of the proof are a result of lifting circle actions to complex line bundles, due to Hattori and Yoshida, and an S^1 -equivariant localization formula, largely due to Bott. Some remarks and related results are also discussed.

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, all manifolds mentioned are connected, smooth, closed and oriented and all circle actions on manifolds are smooth. When we say an almost-complex manifold (M^{2n}, J) and a circle action on it, we mean that the manifold are closed and taken the canonical orientation and the circle action preserves the almost-complex structure J . We use superscripts to denote the *real* dimensions of the manifolds.

In transformation group theory, it is a classical and important topic to study various topological obstructions (such as the vanishing or non-vanishing of certain characteristic numbers) to the existence of non-trivial S^1 -actions on manifolds with specified properties (for example, [2],[6],[13],[15],[16] etc.).

In [8], Fang and Rong provide some restrictions, in terms of the cohomology ring structure, to the existence of fixed point free S^1 -action on manifolds with finite fundamental group. More precisely, they showed ([8], Theorem 1.1)

Theorem 1.1 (Fang-Rong). *Let M^{2n} be a manifold of finite fundamental group. If M admits a fixed point free S^1 -action, then for all $c \in H^2(M; \mathbb{Z})$ and $0 \leq m \leq [\frac{n}{2}]$, we have*

$$(c^{n-2m} \cdot L_m)[M] = 0.$$

In particular, $(c^n)[M] = 0$. Here $[M]$ is the orientation cycle of M and $L = L_0 + L_1 + \cdots + L_{[\frac{n}{2}]}$ is the Hirzebruch L -polynomial of M .

Besides this result, [8] contains several finiteness results on Riemann manifolds with bounded sectional curvature and diameter, of which Theorem 1.1 plays a crucial role in the proofs. The main idea in the proof of Theorem 1.1 is, for any nonnegative integer k , to realize $(e^{kc} \cdot L)[M]$ as the index of the signature operator twisted by some S^1 -equivariant complex line bundle, and then get some constraints on $(c^{n-2m} \cdot L_m)[M]$ by using the equivariant index theorem.

It is well-known that a circle action on an even-dimensional manifold cannot have exactly one fixed point (cf. [6], p.3947). So Theorem 1.1 gives the following implication

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Proposition 1.2. For a manifold M^{2n} of finite fundamental group, if there exists some $c \in H^2(M; \mathbb{Z})$ such that $(c^n)[M]$ is nonzero, then any S^1 -action on M must have at least two fixed points.

Remark 1.3. When a circle action has exactly two fixed points, Ding proved that ([6], p.3947), the tangent spaces at these two fixed points are isomorphic as S^1 -modules and the signature of the manifold vanishes. For \mathbb{Z}_p -actions, there is a similar conclusion under some restrictions ([7]).

Now we recall a result of Pelayo and Tolman in [18]. Suppose (M^{2n}, J) is an almost-complex manifold on which an S^1 -action is given with isolated fixed points, say P_1, P_2, \dots, P_r . At each P_i , there are n integers $k_1^{(i)}, \dots, k_n^{(i)}$ (not necessarily distinct) induced from the isotropy representation of this S^1 -action at P_i . We consider the following r integers:

$$(1.1) \quad \sum_{j=1}^n k_j^{(1)}, \sum_{j=1}^n k_j^{(2)}, \dots, \sum_{j=1}^n k_j^{(r)}.$$

We say this S^1 -action is *somewhere injective* if at least one of the r integers in (1.1) is different from the other $r - 1$ integers. Then we have ([18], Theorem 1)

Theorem 1.4 (Pelayo-Tolman). *Suppose (M^{2n}, J) is an almost-complex manifold on which an S^1 -action is given with isolated fixed points. If this S^1 -action is somewhere injective, then $r \geq n + 1$.*

Remark 1.5. Here r is, of course, the number of the fixed points and equal to the Euler characteristic of (M^{2n}, J) . Although Theorem 1.4 in [18] is formulated for symplectic manifolds and symplectic circle actions. Its proof can be applied directly to almost-complex manifolds.

The main tool in the proof of Theorem 1.4 is an equivariant localization formula, together with a clever trick ([18], Lemma 8). This trick is the origin of the definition of *somewhere injective*.

In [15], inspired by this trick, the present author and Liu, among other things, proved a result ([15], Theorem 1.4), which provides some sufficient conditions to the lower bound of the number of fixed points of S^1 -actions. As a consequence, we have ([15], Corollary 1.5)

Proposition 1.6 (Li-Liu). *If the Chern number $(c_m)^n[M]$ (resp. Pontrjagin number $(p_m)^n[N]$) of an almost-complex manifold (M^{2mn}, J) (resp. smooth manifold N^{4mn}) is nonzero, then any S^1 -action on (M^{2mn}, J) (resp. N^{4mn}) must have at least $n + 1$ fixed points.*

A very special case of this result is, *if the Chern number $(c_1)^n[M]$ of an almost-complex manifold (M^{2n}, J) is nonzero, then any S^1 -action on M^{2n} must have at least $n + 1$ fixed points.*

Recently, we notice that this very special case has been obtained by Hattori in [10] ([9] is an announcement of the main results of [10]). In fact, what Hattori obtained is broader than this and also related to Theorem 1.4. Before stating Hattori's result, let us introduce some notations in [10], which will also be used throughout our paper.

Let M^{2n} be a manifold on which an S^1 -action is given and $c \in H^2(M; \mathbb{Z})$. It is well known that there exists a complex line bundle L over M , which is unique up to bundle isomorphism, such that the first Chern class of L , $c_1(L)$, equals to c . We call c *admissible* if the given S^1 -action can be lifted to an action on the corresponding complex line bundle L . If c is

admissible, then we fix a lifting of the action on L . Moreover, if the given S^1 -action has isolated fixed points, say P_1, \dots, P_r , we consider the fiber $L_{p_i} \cong \mathbb{C}$ of L over the fixed point P_i . Each fiber L_{p_i} is an S^1 -module, and hence corresponds to an integer a_i such that the action of each element $\lambda \in S^1$ on L_{p_i} is given by multiplying λ^{a_i} . The integers a_i will be called the weights of L (with respect to the given lifting) at the fixed point P_i . Note that if we choose another lifting of this S^1 -action then the weights a_i are changed simultaneously to $a_i + a$ for some integer a (see Remark 2.2). So we give the following definition.

Definition 1.7. Suppose c is admissible with respect to an S^1 -action having isolated fixed points. We call c *everywhere injective* if the weights a_1, \dots, a_r are mutually distinct. Similarly, we call such c *somewhere injective* if these a_1, \dots, a_r have the property that, at least one of these integers is different from the other $r - 1$ integers.

Clearly the property of everywhere injective implies that of somewhere injective. Now we can summarize the results of Hattori in [10] related to our paper as follows ([10], p.441-p.443).

Theorem 1.8 (Hattori). *Suppose (M^{2n}, J) is an almost-complex manifold on which an S^1 -action is given, and the fixed points of this action are isolated. We denote by r the number of the fixed points. Let $c \in H^2(M; \mathbb{Z})$ be admissible. We have*

- (1) *If $(c^n)[M] \neq 0$, then $r \geq n + 1$.*
- (2) *If c is everywhere injective, then $r \geq n + 1$.*
- (3) *If $r = n + 1$, then the condition $(c^n)[M] \neq 0$ holds if and only if c is everywhere injective.*

The main purpose of present paper is, by using a special case of the equivariant localization formulae, largely due to Bott, and a result of lifting circle actions to complex line bundles, due to Hattori and Yoshida, to unify Proposition 1.2, Theorem 1.4 and Theorem 1.8 into the following theorem.

Theorem 1.9. *Suppose M^{2n} is a smooth manifold on which an S^1 -action is given, and $c \in H^2(M; \mathbb{Z})$ is admissible.*

- (1) *If $(c)^n[M] \neq 0$, then the given action has at least $n + 1$ fixed points.*
- (2) *If the given action has isolated fixed points and c is somewhere injective, then the number of the fixed points is no less than $n + 1$.*
- (3) *Suppose the given circle action has exactly $n + 1$ fixed points. Then there are precisely two possibilities:*
 - (a) *$(c)^n[M] \neq 0$ and c is everywhere injective;*
 - (b) *$(c)^n[M] = 0$ and c is not somewhere injective.*

Corollary 1.10. Suppose M^{2n} is a smooth manifold admitting an S^1 -action with exactly $n + 1$ fixed points. If an admissible c ($\in H^2(M; \mathbb{Z})$) is somewhere injective, it must be everywhere injective.

Clearly Theorem 1.8 can be derived from Theorem 1.9. Now let us see how to generalize Proposition 1.2 and Theorem 1.4 from Theorem 1.9.

In the next section we will see that (Remark 2.2), for a manifold M with vanishing first Betti number $b_1(M)$, any element in $H^2(M; \mathbb{Z})$ is admissible with respect to any circle action on M . Therefore we have the following corollary, which is a generalization of Proposition 1.2.

Corollary 1.11. For a manifold M^{2n} with $b_1(M) = 0$, if there exists some $c \in H^2(M; \mathbb{Z})$ such that $(c^n)[M]$ is nonzero, then any S^1 -action on M must have at least $n + 1$ fixed points.

The following corollary is parallel to ([8], Theorem 1.2).

Corollary 1.12. Let M^{2n} be a symplectic manifold with vanishing first Betti number. Then any *smooth* circle action on M must have at least $n + 1$ fixed points.

Remark 1.13. When the action is symplectic, then it must be Hamiltonian as $b_1(M) = 0$. In this case the corollary is well-known because of the existence of the moment map (a perfect Morse-Bott function) and the Morse inequality (cf. the fourth paragraph of [18]).

In order to derive Theorem 1.4 from Theorem 1.9, let us consider the first Chern class $c_1(M, J)$ of an almost-complex manifold (M^{2n}, J) . It is also the first Chern class of the complex line bundle $\bigwedge^n(TM)$ over M by taking the n -th wedge product of TM . Here TM is the holomorphic tangent bundle of M in the sense of J . So $c_1(M, J)$ is admissible with respect to any S^1 -action on M . If (M^{2n}, J) has an S^1 -action with isolated fixed points P_1, P_2, \dots, P_r . Then at each P_i , we have the weights $k_1^{(i)}, \dots, k_n^{(i)}$ induced from the S^1 -module $T_{P_i}M$. Clearly the weight of this action at the fiber $\bigwedge^n(TM)|_{P_i} = \bigwedge^n(T_{P_i}M)$ is $\sum_{j=1}^n k_j^{(i)}$. Then Theorem 1.4 follows from the second assertion of Theorem 1.9.

2. PRELIMINARIES

2.1. Lifting circle actions to line bundles. Let a compact Lie group G act on a manifold M and E be a fiber bundle over M . It is natural to ask whether, when, and how the G -action on M can be lifted to a G -action on E . In general this is a difficult problem. But in some cases, it has been solved completely. For example, in [11], Hattori and Yoshida solved the case of principal torus bundles (hence of complex line bundles as a special case).

Let $ES^1 \rightarrow BS^1$ be the universal S^1 -bundle. Given an S^1 -action, we have the fibration

$$M \xrightarrow{i} M_{S^1} \rightarrow BS^1,$$

where $M_{S^1} = ES^1 \times_{S^1} M$ is the Borel construction.

The following proposition is due to Hattori and Yoshida ([11], cf. [10], p.440), which is summarized to fit into our applications.

Proposition 2.1 (Hattori-Yoshida). Let $L \rightarrow M$ be a complex line bundle. The given S^1 -action of M can be lifted to a linear action on L if and only if

$$c_1(L) \in i^*H^2(M_{S^1}; \mathbb{Z}).$$

Furthermore, if $c_1(L) \in i^*H^2(M; \mathbb{Z})$, all different liftings are parametrized by $H^2(BS^1; \mathbb{Z}) \cong \mathbb{Z}$.

Remark 2.2. (1) To the author's best knowledge, this proposition was proved again by several people ([17], Theorem 1.1), ([6], Corollary 2.2)).

(2) The most useful case is, if $b_1(M) = 0$, then any element in $H^2(M; \mathbb{Z})$ is admissible with respect to any S^1 -action on M ([10], Lemma 3.2), ([6], Corollary 2.2)). That is why we have Corollary 1.11.

(3) The fact that, if can, then different liftings of a given S^1 -action on M to a complex line bundle L is parametrized by \mathbb{Z} , has an important implication: when the S^1 -action has isolated fixed points, say P_1, \dots, P_r , any chosen lifting to L has corresponding weights

a_i at P_i ; if we choose another lifting then the weights a_i are changed simultaneously to $a_i + a$ for some integer a . This observation will play a crucial role in the proof of our main result (see Lemma 3.1).

2.2. Localization formula for S^1 -equivariant complex vector bundles. What we need in this paper is the localization formula for S^1 -equivariant complex line bundle. For completeness, we review it for complex vector bundles.

Let E be a complex vector bundle over a smooth manifold M^{2n} with fiber (complex) dimension m . Suppose we have an S^1 -action on M which can be lifted to E , and we select a lifting. For simplicity, we assume the fixed points of this action be isolated, say P_1, \dots, P_r as before. At each P_i , each fiber E_{P_i} is a m -dimensional complex S^1 -module induced from the action, to which n integer weights $a_1^{(i)}, \dots, a_m^{(i)}$ are attached. The tangent space $T_{P_i}M$ at P_i is a $2n$ -dimensional real S^1 -module also induced from the action, and splits as follows

$$T_{P_i}M = \bigoplus_{j=1}^n V_j^{(i)},$$

where each $V_j^{(i)}$ is a real 2-plane. We can choose an isomorphism of \mathbb{C} with $V_j^{(i)}$ relative to which the representation of S^1 on $V_j^{(i)}$ is given by $\lambda \mapsto \lambda^{k_j^{(i)}}$ with $k_j^{(i)} \in \mathbb{Z} - \{0\}$. We further assume the weights $k_1^{(i)}, \dots, k_n^{(i)}$ be chosen in such a way that the usual orientations on the summands $V_j^{(i)} \cong \mathbb{C}$ induce the given orientation on $T_{P_i}M$. Note that these $k_1^{(i)}, \dots, k_n^{(i)}$ are uniquely defined up to even number of sign changes. In particular, their product $\prod_{j=1}^n k_j^{(i)}$ is well-defined.

Let $f(x_1, \dots, x_m)$ be a symmetric polynomial in the variables x_1, \dots, x_m . Then $f(x_1, \dots, x_m)$ can be written in a unique way in terms of the elementary symmetric polynomials $\tilde{f}(e_1, \dots, e_m)$, where $e_i = e_i(x_1, \dots, x_m)$ is the i -th elementary symmetric polynomial of x_1, \dots, x_m .

Now we can state a localization formula, largely due to Bott, which reduces the computations of Chern numbers of the S^1 -equivariant complex vector bundle E to $\{k_j^{(i)}\}$ and $\{a_j^{(i)}\}$, as follows.

Theorem 2.3 (localization formula). *With above notations understood and moreover suppose the degree of $f(x_1, \dots, x_m)$ is n ($\deg(x_i) = 1$). Then*

$$(2.1) \quad \tilde{f}(c_1, \dots, c_m) \cdot [M] = \sum_{i=1}^r \frac{f(a_1^{(i)}, \dots, a_m^{(i)})}{\prod_{j=1}^n k_j^{(i)}},$$

where c_i is the i -th Chern class of E .

Remark 2.4. (1) If the action has no fixed points (though not necessarily free) it follows that all Chern numbers are zero.

(2) When M is an almost-complex manifold and E is the holomorphic tangent bundle (in the sense of J), this formula is essentially due to Bott ([4], [5] and ([3], §6)). This general form can be derived from Atiyah-Bott equivariant localization formula in [1]. In particular, when E is a line bundle, which is what we only need in this paper, it was carried out explicitly in ([1], §8, (8.13)).

- (3) If the lifting to E varies, then so is $a_j^{(i)}$. But the Chern numbers depend only on the underlying complex vector bundle E . Thus (2.1) provides certain constraints on $\{a_j^{(i)}\}$ and $\{k_j^{(i)}\}$.
- (4) When M is an almost-complex manifold and E is the holomorphic tangent bundle, (2.1) is also valid for those f whose degrees are less than n (in this case the left-side hand of (2.1) is zero by definition). But for general E , this is *not* the case (cf. the footnote on p.313 of [5]).

3. PROOF OF THEOREM 1.9

Note that in the first assertion of Theorem 1.9 we do not require the fixed points of the action to be isolated. The reason is the following: if $(c^n)[M] \neq 0$, then the S^1 -action must have non-empty fixed points by remark 2.4. If the fixed point set is not isolated, then at least one connected component is a submanifold of positive dimension. In this case there are infinitely many fixed points. Thus in order to prove the first assertion of Theorem 1.9, it suffices to consider those S^1 -actions with isolated fixed points.

So in this section, we assume that M^{2n} admits an S^1 -action, which can be lifted to a complex line bundle L , and the fixed points are P_1, \dots, P_r . At each P_i , we have the weights $k_1^{(i)}, \dots, k_n^{(i)}$ induced from the S^1 -module $T_{P_i}M$. We fix a lifting of this action and therefore we have weight a_i at P_i induced from the S^1 -module L_{P_i} .

As remarked in Remark 2.4, in general, (2.1) does not hold for those f whose degrees are less than n . But the following lemma, which is inspired by a conversation with Prof. Paul Baum, tells us that, for line bundles, this is the case.

Lemma 3.1. *We have the following constraints on $\{k_j^{(i)}\}$ and $\{a_i\}$.*

$$\sum_{i=1}^r \frac{(a_i)^t}{\prod_{j=1}^n k_j^{(i)}} = 0, \quad t = 0, 1, \dots, n-1.$$

Proof. Applying (2.1) we have

$$(3.1) \quad (c_1(L))^n[M] = \sum_{i=1}^r \frac{(a_i)^n}{\prod_{j=1}^n k_j^{(i)}}.$$

We have mentioned in Remark 2.2 that, if we choose another lifting of this S^1 -action to L , then the weights a_1, \dots, a_n are changed simultaneously to $a_1 + a, \dots, a_n + a$ for some $a \in \mathbb{Z}$. But different liftings are parametrized by \mathbb{Z} , which means

$$(3.2) \quad (c_1(L))^n[M] = \sum_{i=1}^r \frac{(a_i + a)^n}{\prod_{j=1}^n k_j^{(i)}}, \quad \forall a \in \mathbb{Z}.$$

Comparing (3.1) to (3.2), we deduce that

$$\sum_{t=0}^{n-1} \left[\binom{n}{t} \cdot a^{n-t} \cdot \left(\sum_{i=1}^r \frac{(a_i)^t}{\prod_{j=1}^n k_j^{(i)}} \right) \right] = 0, \quad \forall a \in \mathbb{Z}.$$

Therefore,

$$\sum_{i=1}^r \frac{(a_i)^t}{\prod_{j=1}^n k_j^{(i)}} = 0, \quad t = 0, 1, \dots, n-1.$$

□

Let

$$\{a_i \mid 1 \leq i \leq r\} = \{s_1, \dots, s_l\} \subset \mathbb{Z}.$$

Clearly, s_1, \dots, s_l are mutually distinct, $l \leq r$, $l = r$ if and only if $c_1(L)$ is everywhere injective. Define

$$A_t := \sum_{\substack{1 \leq i \leq r \\ a_i = s_t}} \frac{1}{\prod_{j=1}^n k_j^{(i)}}, \quad 1 \leq t \leq l.$$

Note that $c_1(L)$ is somewhere injective only if at least one of these A_t is nonzero.

The first two assertions in Theorem 1.9 will be derived from the following two lemmas.

Lemma 3.2. *If $c_1(L)$ is somewhere injective or $(c_1(L))^n[M] \neq 0$, then at least one of A_t is nonzero.*

Proof. If $c_1(L)$ is somewhere injective, then, by definition, some s_t coincides with exactly one of a_1, \dots, a_r . Hence the corresponding A_t has only one term $\frac{1}{\prod_{j=1}^n k_j^{(i)}}$, which is of course nonzero.

As for the second assertion, it easily follows from (2.1) that

$$(c_1(L))^n[M] = \sum_{t=1}^l (s_t)^n \cdot A_t.$$

□

The idea of the following lemma was first used by Pelayo-Tolman ([18], Lemma 8), then by the present author and Liu ([15], Lemma 3.2).

Lemma 3.3. *If r , the number of the fixed points, is no more than n , then $A_t = 0$ for all $t = 1, \dots, l$.*

Proof. By virtue of Lemma 3.1 we have

$$(3.3) \quad \begin{cases} A_1 + A_2 + \dots + A_l = 0 \\ s_1 A_1 + s_2 A_2 + \dots + s_l A_l = 0 \\ \vdots \\ (s_1)^{n-1} A_1 + (s_2)^{n-1} A_2 + \dots + (s_l)^{n-1} A_l = 0 \end{cases}$$

Note that $l \leq r$ and so by assumption $l \leq n$. By definition s_1, \dots, s_l are mutually distinct, which means the coefficient matrix of the first l lines of (3.3) is nonsingular (so-called Vandermonde matrix). Thus the only possibility is

$$A_1 = \dots = A_l = 0.$$

□

The last lemma will lead to the proof of the third assertion in Theorem 1.9.

Lemma 3.4. *Suppose $r = n + 1$.*

- (1) *If $l \leq r - 1 = n$, then $A_1 = \cdots = A_l = 0$. Consequently, $(c_1(L))^n[M] = 0$ and $c_1(L)$ is not somewhere injective.*
- (2) *If $l = r = n + 1$, then $c_1(L)$ is everywhere injective and $(c_1(L))^n[M] \neq 0$.*

Proof. (1) follows from (3.3).

For (2), $c_1(L)$ is everywhere injective as $l = r$. If $(c_1(L))^n[M] = 0$, then

$$(3.4) \quad (s_1)^n A_1 + (s_2)^n A_2 + \cdots + (s_l)^n A_l = 0.$$

Combining (3.4), (3.3) and $l = n + 1$ we deduce that $A_1 = \cdots = A_l = 0$, which means $c_1(L)$ is not somewhere injective, a contradiction. \square

4. CONCLUDING REMARKS

4.1. A note on a result of Hattori. In ([10], p.449) Hattori proved this proposition: *if (M^{2n}, J) is an almost-complex manifold admitting an S^1 -action with isolated fixed points, and the first Chern class, $c_1(M)$, is a torsion element, then $Td(M)[M] = 0$.* Here $Td(M) \in H^*(M; \mathbb{Q})$ is the Todd class ([12]).

In fact, by using a celebrated theorem of Atiyah and Hirzebruch in [2], we have the following proposition.

Proposition 4.1. Let (M^{2n}, J) be an almost-complex manifold admitting a non-trivial S^1 -action. If $c_1(M)$ is a torsion element such that $c_1(M) \equiv 0 \pmod{2}$, then

$$Td(M)[M] = 0.$$

Proof. Suppose the total Chern class of (M^{2n}, J) has the following formal factorization:

$$1 + c_1(M) + \cdots + c_n(M) = \prod_{i=1}^n (1 + x_i),$$

i.e., $c_i(M)$ is the i -th elementary symmetric polynomial of x_1, \dots, x_n . Then

$$Td(M) = \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}}, \quad \hat{A}(M) = \prod_{i=1}^n \frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} = e^{-\frac{c_1(M)}{2}} Td(M).$$

Here $\hat{A}(M) \in H^*(M; \mathbb{Q})$ is the \hat{A} -class. When M is a spin manifold, $\hat{A}(M)[M]$ is the index of the Dirac operator ([3], §5). Now

$$Td(M)[M] = (e^{\frac{c_1(M)}{2}} \hat{A}(M))[M].$$

If $c_1(M)$ is a torsion element, then

$$Td(M)[M] = \hat{A}(M)[M].$$

The assumption of mod 2 reduction of $c_1(M)$ being zero implies that M is a spin manifold. A celebrated theorem of Atiyah and Hirzebruch asserts that, for a spin manifold N , the index of the Dirac operator on N vanishes if N admits a non-trivial S^1 -action. This completes the proof of this proposition \square

Remark 4.2. As pointed out in ([10], p.449), when n is odd, $Td(M^{2n})$ is of the form $c_1(M) \cdot h$ ($h \in H^*(M; \mathbb{Q})$) ([12]). Thus $Td(M) = 0$ when $c_1(M)$ is a torsion element. So this proposition only makes sense when n is even.

4.2. Final remarks. The original purpose of [10] was to investigate the constraints on certain circle actions which have similar feature to those arising from the positivity of curvature, as mentioned by its author in the Introduction. But [10] contains many interesting results which are of independent interest and deserves further attention.

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DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI 200092, CHINA

E-mail address: pingli@tongji.edu.cn